Exercise 8

Use power series to solve the differential equation.

$$y'' = xy$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution centered here.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Differentiate the series with respect to x once more.

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these formulas into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = x \sum_{n=0}^{\infty} a_n x^n$$

Bring x inside the summand.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^{n+1}$$

Make the substitution $n = k + 3$ in the series on the left and the substitution $n = k$ in the series on the right.

$$\sum_{k+3=2}^{\infty} (k+3)[(k+3)-1] a_{k+3} x^{(k+3)-2} = \sum_{k=0}^{\infty} a_k x^{k+1}$$

Simplify the left side.

$$\sum_{k=-1}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1} = \sum_{k=0}^{\infty} a_k x^{k+1}$$

Write out the first term on the left side.

$$2a_2 + \sum_{k=0}^{\infty} (k+3)(k+2) a_{k+3} x^{k+1} = \sum_{k=0}^{\infty} a_k x^{k+1}$$

Combine the series.

$$2a_2 + \sum_{k=0}^{\infty} \left[(k+3)(k+2) a_{k+3} - a_k\right] x^{k+1} = 0$$

a_2 and the quantity in square brackets must be zero.

$$(k+3)(k+2)a_{k+3} - a_k = 0 \qquad a_2 = 0$$

Solve for a_{k+3}.

$$a_{k+3} = \frac{1}{(k+3)(k+2)} a_k$$

In order to determine a_k, plug in values for k and try to find a pattern.

$$k = 0: \quad a_3 = \frac{1}{(0+3)(0+2)} a_0 = \frac{1}{3 \cdot 2} a_0$$

$$k = 1: \quad a_4 = \frac{1}{(1+3)(1+2)} a_1 = \frac{1}{4 \cdot 3} a_1$$

$$k = 2: \quad a_5 = \frac{1}{(2+3)(2+2)} a_2 = 0$$

$$k = 3: \quad a_6 = \frac{1}{(3+3)(3+2)} a_3 = \frac{1}{6 \cdot 5} \left(\frac{1}{3 \cdot 2} a_0 \right) = \frac{1}{6 \cdot 5 \cdot 3 \cdot 2} a_0$$

$$k = 4: \quad a_7 = \frac{1}{(4+3)(4+2)} a_4 = \frac{1}{7 \cdot 6} \left(\frac{1}{4 \cdot 3} a_1 \right) = \frac{1}{7 \cdot 6 \cdot 4 \cdot 3} a_1$$

$$k = 5: \quad a_8 = \frac{1}{(5+3)(5+2)} a_5 = 0$$

$$\vdots$$

The general formula is

$$a_{3m} = \frac{a_0}{(3m)(3m-1)(3m-3)(3m-4)\cdots 9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}$$

$$a_{3m+1} = \frac{a_1}{(3m+1)(3m)(3m-2)(3m-3)\cdots 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}$$

$$a_{3m+2} = 0.$$

Therefore, the general solution is

$$
\begin{aligned}
y(x) &= \sum_{m=0}^{\infty} a_m x^m \\
&= \sum_{m=0}^{\infty} a_{3m} x^{3m} + \sum_{m=0}^{\infty} a_{3m+1} x^{3m+1} + \sum_{m=0}^{\infty} a_{3m+2} x^{3m+2} \\
&= \sum_{m=0}^{\infty} \frac{a_0}{(3m)(3m-1)(3m-3)(3m-4)\cdots 9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} x^{3m} \\
&\qquad + \sum_{m=0}^{\infty} \frac{a_1}{(3m+1)(3m)(3m-2)(3m-3)\cdots 9\cdot 7\cdot 6\cdot 4\cdot 3} x^{3m+1} + \sum_{m=0}^{\infty} (0) x^{3m+2} \\
&= a_0 \sum_{m=0}^{\infty} \frac{x^{3m}}{(3m)(3m-1)(3m-3)(3m-4)\cdots 9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} \\
&\qquad + a_1 \sum_{m=0}^{\infty} \frac{x^{3m+1}}{(3m+1)(3m)(3m-2)(3m-3)\cdots 9\cdot 7\cdot 6\cdot 4\cdot 3},
\end{aligned}
$$

where a_0 and a_1 are arbitrary constants.